October Math Gems

Problem of the week 4

§1 problems

Problem 1.1. In the figure below ABC is a right-angled triangle and BD an angle bisector. If $AB = 3$, and the area of $ABD = 9$, what is the length of DC?

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Problem 1.2. If a, b, c are positive reals with $abc = 1$. what is the minimum value of

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(a+c)} + \frac{1}{c^3(a+b)}$$

?

Problem 1.3. How many ordered pairs of positive integers (M, N) satisfy the equation $\frac{M}{6} = \frac{6}{N}$?

Problem 1.4. Two median are drawn from acute angles a right angled triangle intersect at an angle $\frac{\pi}{6}$. If the length of the hypotenuse of the triangle $= 3$, then, find the area of triangle=

Problem 1.5. If the points $a(3,4), b(7,12), andp(x,y)$ are such that $(pa^2 > (pb)^2 > (ab^2))$ Evaluate x where x is integral number.

Problem 1.6. Known that $a+b+c=\pi$, then

$$\frac{\sin 2a+\sin 2b+\sin 2c}{\cos a+\cos b+\cos c-1}=$$

Problem 1.7. Given $g(x)=9\log_8(x-3)-5$, $g^{-1}(13)=$

Problem 1.8. let $x,y,z>0$. Prove that

$$\frac{2}{x+y}+\frac{2}{y+z}+\frac{2}{z+x}\geq\frac{9}{x+y+z}$$

Problem 1.9. Determine the domain of the function

$$g(x)=\cot^{-1}\left(\frac{x}{\sqrt{x^2-\lfloor x^2\rfloor}}\right)$$

Problem 1.10. Let α,β, and γ denote the angles of a triangle. Show that

$$\sin\alpha+\sin\beta+\sin\gamma=4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2},$$

$$\sin 2\alpha+\sin 2\beta+\sin 2\gamma=4\sin\alpha\sin\beta\sin\gamma$$

$$\sin 4\alpha+\sin 4\beta+\sin 4\gamma=-4\sin 2\alpha\sin 2\beta\sin 2\gamma.$$

Problem 1.11. Prove that no number in the sequence

$$11,111,1111,11111,\ldots.$$

is the square of an integer.

Problem 1.12. Solve the following system of three equations for the unknowns x, y and z :

$$\begin{aligned}5732x+2134y+2134z&=7866\\2134x+5732y+2134z&=670\\2134x+2134y+5732z&=11464\end{aligned}$$

Problem 1.13. A pyramid is called "regular" if its base is a regular polygon and the foot of its altitude is the center of its base. A regular pyramid has a hexagonal base the area of which is one quarter of the total surface-area S of the pyramid. The altitude of the pyramid is h. Express S in terms of h.

Problem 1.14. Let a and b be positive real numbers satisfying

$$\frac{a}{b}\left(\frac{a}{b}+2\right)+\frac{b}{a}\left(\frac{b}{a}+2\right)=2022.$$

Find the positive integer n such that

$$\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}=\sqrt{n}$$

Problem 1.15. Solve

$$x^{x^{x^{2021}}} = 2021$$

Problem 1.16. If α, β, γ do not differ by a multiple of π and if

$$\frac{\cos(\alpha+\theta)}{\sin(\beta+\gamma)} = \frac{\cos(\beta+\theta)}{\sin(\gamma+\alpha)} = \frac{\cos(\gamma+\theta)}{\sin(\alpha+\beta)} = K$$

Find the value of K.

Problem 1.17. If $x = \sqrt{3\sqrt{2\sqrt{3\sqrt{2\sqrt{3\sqrt{2...}}}}}}$, Find the value of x^2

Problem 1.18. Solve for x

$$\sqrt[4]{1-x^2} + \sqrt[4]{1-x} + \sqrt[4]{1+x} = 3$$

Problem 1.19. There are real numbers a, b, c, and d such that -20 is a root of x^3+ax+b and -21 is a root of x^3+cx^2+d. These two polynomials share a complex root $m+\sqrt{n}\cdot i$, where m and n are positive integers and $i=\sqrt{-1}$. Find $m+n$.

Problem 1.20. Given that

$$a + \frac{3}{b} = 3$$

$$b + \frac{2}{c} = 2$$

$$c + \frac{1}{a} = 1$$

Find $a+2b+3c$